

Coherent states for continuous spectrum operators with non-normalizable fiducial states

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Abstract

The problem of building coherent states from non-normalizable fiducial states is considered. We propose a way of constructing such coherent states by regularizing the divergence of the fiducial state norm. Then, we successfully apply the formalism to particular cases involving systems with a continuous spectrum: coherent states for the free particle and for the inverted oscillator ($p^2 - x^2$) are explicitly provided. Similar ideas can be used for other systems having non-normalizable fiducial states.

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1 Introduction

Coherent states are acknowledged to be essential objects in understanding the connection between the quantum and the classical counterparts of a system [1]-[11]. Furthermore, they have become of significance in more formal and interesting mathematical developments [2][3][7][9]. In fact, coherent states can be constructed for any formal Hilbert space, having either a discrete or continuous basis, including those related to, for instance, a quantum mechanical system [8]. Note that the problem of continuous spectrum dynamical operators and the construction of their associated coherent states have been addressed in different fashions but, generally most can be recast in the formulation of Gazeau-Klauder (GK) [8]. Besides, in the latter work, the authors have provided an axiomatic approach for families of states in order to be called *physical* coherent states. The GK axioms are simple to describe: continuity in the labels, a resolution of the identity of the Hilbert space, temporal stability of states and the action identity. This approach has led to other developments on continuous spectrum operators. For instance, it turns out that ladder operators for continuous spectrum operators can be defined and corresponding eigenvalue problems for defining coherent states can be explicitly solved [12]. Statistical properties and other nonclassical properties of such states have been recently addressed [13].

Coherent states may be constructed in three different ways: they can be solutions of an eigenvalue problem of an annihilation operator; they can be constrained to satisfy some uncertainty principle or, finally, they can be defined as orbits of a unitary operator acting on a preferred or *fiducial* state. In the continuous spectrum case, it might happen that the eigenstates of the Hamiltonian are all non-normalizable. Focusing on the last proposal, the family of states generated by a unitary transformation of any non-normalizable fiducial state will be obviously non-normalizable and so would not satisfy elementary properties of coherent states. Clearly, the problem of non-normalizable states is recurrent when one is dealing with eigenstates of continuous spectrum operators. Building coherent states for non-normalizable fiducial states has already been considered in the literature and solved with different techniques according to the particular model under consideration [14, 15, 16, 11], and it still attracts attention [17]. We provide, in the present work, a specific formulation to address this issue of non-normalizable fiducial states and how to define coherent states based on them.

In this paper, we consider the construction of coherent states for dynamical systems with continuous spectrum having non-normalizable states. Two ways of regularizing the fiducial state are introduced and, for these choices, two families of coherent states can be built in the canonical quantization framework. We also investigate the main properties of the coherent states with respect to the GK axioms. We find that the two coherent state families are indeed continuous in labels, normalized and satisfy a resolution of the identity on the Hilbert space. The remaining GK axioms has to be checked case by case. Explicit examples are treated for the free particle coherent states and the so-called inverted harmonic oscillator [14]. Interestingly, we show that in a leading approximation in the regularization parameter, coherent states of these two examples are stable under time evolution.

The paper is organized as follows: the next section is devoted to the general formulation of canonical coherent states with the regularized fiducial states. Section 3 consists in the applications of our formalism: first the free particle and, then, the inverted harmonic oscillator

are investigated. Section 4 gives a summary of our results and an appendix provides useful identities and other details for the text.

2 Coherent states and non-normalizable fiducial states

In the present section, we formally discuss a way of constructing coherent states based on a non-normalizable fiducial vector. This vector is selected among the eigenbasis spanning the Hilbert space related with the spectral decomposition of a given Hamiltonian operator. Although our study is tailored to operators of the continuous spectrum type, this does not preclude a natural extension of the formulation to the continuous and discrete spectrum situations.

Let us consider a Hamiltonian operator H with a continuous energy spectrum and let $\{|\Psi_E\rangle\}$ be the family of eigenvectors of H , i.e.

$$H|\Psi_E\rangle = E|\Psi_E\rangle, \quad E \in \mathbb{R}, \quad (1)$$

such that $\{|\Psi_E\rangle\}_{E \in \mathbb{R}}$ forms an orthogonal basis, labelled continuously by E , of vectors of an abstract Hilbert space \mathcal{H} associated with H . Note that we do not exclude the case of negative spectrum, but will require, as usual, that the spectrum of H is real. We also assume that

$$\langle \Psi_E | \Psi_{E'} \rangle = \delta(E - E'), \quad \langle \Psi_E | \Psi_E \rangle = \infty. \quad (2)$$

Since the states $|\Psi_E\rangle$ are not normalizable, none of them can be chosen as a preferred vector for building normalized coherent states. Two kinds of *regularized* fiducial vectors $|\eta_{\bar{E}}\rangle$ and $|\eta_{\bar{E}}\rangle_A$ will be constructed by a weighted integral of the states $|\Psi_E\rangle$ on an interval of energy as follows:

$$(a) \quad |\eta_{\bar{E}}\rangle = C \int_{E_0}^{E_1} |\Psi_E\rangle dE; \quad \langle \eta_{\bar{E}} | \eta_{\bar{E}} \rangle = 1 \quad \Rightarrow \quad C = \frac{1}{\sqrt{E_1 - E_0}}; \quad (3)$$

$$(b) \quad |\eta_{\bar{E}}\rangle_A = C_A \int_{-\infty}^{+\infty} e^{-A(E - \bar{E})^2} |\Psi_E\rangle dE, \quad A \gg 1; \\ A \langle \eta_{\bar{E}} | \eta_{\bar{E}} \rangle_A = 1 \quad \Rightarrow \quad C_A = \left(\frac{2A}{\pi} \right)^{\frac{1}{4}}. \quad (4)$$

Here \bar{E} is the mean of the weight function which in the uniform case is $(E_1 - E_0)/2$. Observe that the new vectors $|\eta_{\bar{E}}\rangle$ and $|\eta_{\bar{E}}\rangle_A$ are now normalized to 1. It is noteworthy that the regularization procedure introduced in (3) follows that of Isham and Klauder on coherent states for the Euclidean group $E(n)$ [11]. Our remaining goal is to construct two families of vectors using $|\eta_{\bar{E}}\rangle$ and $|\eta_{\bar{E}}\rangle_A$ satisfying a normalizability condition and possessing a resolution of identity of the Hilbert space \mathcal{H} . These two requirements belong to the set of GK physical axioms [8] for coherent states. All GK axioms may not be implemented *a priori* for the states that one can define from $|\eta_{\bar{E}}\rangle$ or $|\eta_{\bar{E}}\rangle_A$; however, that can be remedied later in the analysis.

The definition of coherent states by applying a unitary operator on the fiducial state will be considered. In the following, we will focus on the ordinary canonical phase space

quantization, but the same ideas and appropriate coherent states can be introduced for different types of systems using the general group theoretical formalism [3].

Let us assume that an ordinary canonical quantization procedure leads to the quantum Hamiltonian H . On a $2M$ dimensional phase space, one introduces a set of canonical commutation relations between coordinate operators $[Q_k, P_l] = i\hbar\delta_{kl}$, $k, l = 1, 2, \dots, M$. There exists a special class of phase space unitary operators defined such that $U(\mathbf{q}, \mathbf{p}) = \prod_{\ell=1}^M e^{-\frac{i}{\hbar}q_\ell P_\ell} e^{\frac{i}{\hbar}p_\ell Q_\ell}$, with q_ℓ and p_ℓ some real parameters, $\mathbf{q} = \{q_\ell\}$ and $\mathbf{p} = \{p_\ell\}$. These unitary operators will act on the fiducial states (3) and (4), respectively, as

$$|\mathbf{q}, \mathbf{p}; \bar{E}\rangle = U(\mathbf{q}, \mathbf{p})|\eta_{\bar{E}}\rangle \quad (5)$$

$$|\mathbf{q}, \mathbf{p}; \bar{E}\rangle_A = U(\mathbf{q}, \mathbf{p})|\eta_{\bar{E}}\rangle_A \quad (6)$$

in order to define new families of states. These two families of states fulfill the following:

(i) Normalization condition

$$\langle \mathbf{q}, \mathbf{p}; \bar{E} | \mathbf{q}, \mathbf{p}; \bar{E} \rangle = \langle \eta_{\bar{E}} | \eta_{\bar{E}} \rangle = 1 ; \quad {}_A \langle \mathbf{q}, \mathbf{p}; \bar{E} | \mathbf{q}, \mathbf{p}; \bar{E} \rangle_A = {}_A \langle \eta_{\bar{E}} | \eta_{\bar{E}} \rangle_A = 1 ; \quad (7)$$

(ii) Continuity in labels: As $\|\mathbf{q} - \mathbf{q}'\|_{\mathbb{R}^M} \rightarrow 0$ and $\|\mathbf{p} - \mathbf{p}'\|_{\mathbb{R}^M} \rightarrow 0$

$$\begin{aligned} \|\mathbf{q}, \mathbf{p}; \bar{E}\rangle - \mathbf{q}', \mathbf{p}'; \bar{E}\rangle\|_{\mathcal{H}}^2 &= 2 - 2\Re \langle \mathbf{q}, \mathbf{p}; \bar{E} | \mathbf{q}', \mathbf{p}'; \bar{E} \rangle \rightarrow 0 , \\ \|\mathbf{q}, \mathbf{p}; \bar{E}\rangle_A - \mathbf{q}', \mathbf{p}'; \bar{E}\rangle_A\|_{\mathcal{H}}^2 &= 2 - 2\Re {}_A \langle \mathbf{q}, \mathbf{p}; \bar{E} | \mathbf{q}', \mathbf{p}'; \bar{E} \rangle_A \rightarrow 0 , \end{aligned} \quad (8)$$

where we used the fact that the unitary operators are weakly continuous in their labels.

(iii) Resolution of identity: Introducing the eigenbasis of the position operators $|x_k\rangle$, such that $Q_k|x_k\rangle = x_k|x_k\rangle$, we consider a general tensor product of states $|\mathbf{x}\rangle = \otimes_k |x_k\rangle$, such that, we evaluate

$$\begin{aligned} \langle \mathbf{x} | \mathbf{x}' \rangle &= \int_{\mathbb{R}^M \times \mathbb{R}^M} \langle \mathbf{x} | \mathbf{q}, \mathbf{p}; \bar{E} \rangle \langle \mathbf{q}, \mathbf{p}; \bar{E} | \mathbf{x}' \rangle \frac{d\mathbf{q}d\mathbf{p}}{K} , \\ \langle \mathbf{x} | \mathbf{x}' \rangle &= \int_{\mathbb{R}^M \times \mathbb{R}^M} \langle \mathbf{x} | \mathbf{q}, \mathbf{p}; \bar{E} \rangle_A {}_A \langle \mathbf{q}, \mathbf{p}; \bar{E} | \mathbf{x}' \rangle \frac{d\mathbf{q}d\mathbf{p}}{K} , \end{aligned} \quad (9)$$

for some constant K . Focusing on the first integral, we have

$$\begin{aligned} \langle \mathbf{x} | \mathbf{x}' \rangle &= K^{-1} \int_{\mathbb{R}^M \times \mathbb{R}^M} \langle \mathbf{x} | U(\mathbf{q}, \mathbf{p}) | \eta_{\bar{E}} \rangle \langle \eta_{\bar{E}} | U^\dagger(\mathbf{q}, \mathbf{p}) | \mathbf{x}' \rangle d\mathbf{q}d\mathbf{p} \\ &= (2\pi\hbar)^M K^{-1} \delta^{(M)}(\mathbf{x} - \mathbf{x}') \int_{\mathbb{R}^M} \eta_{\bar{E}}(\mathbf{x} - \mathbf{q}) (\eta_{\bar{E}}(\mathbf{x} - \mathbf{q}))^* d\mathbf{q} \\ &= (2\pi\hbar)^M K^{-1} \delta^{(M)}(\mathbf{x} - \mathbf{x}') \int_{\mathbb{R}^M} \eta_{\bar{E}}(\mathbf{q}) (\eta_{\bar{E}}(\mathbf{q}))^* d\mathbf{q} \end{aligned} \quad (10)$$

as follows from a simple change of variable. Setting $K = (2\pi\hbar)^M$, one gets

$$\int_{\mathbb{R}^M \times \mathbb{R}^M} \langle \mathbf{x} | \mathbf{q}, \mathbf{p}; \bar{E} \rangle \langle \mathbf{q}, \mathbf{p}; \bar{E} | \mathbf{x}' \rangle \frac{d\mathbf{q}d\mathbf{p}}{(2\pi\hbar)^M} = \langle \mathbf{x} | \mathbf{x}' \rangle \quad (11)$$

proving that the states $|\mathbf{q}, \mathbf{p}; \bar{E}\rangle$ obey a resolution of the identity. The proof of the resolution of the identity for the second set of vectors (6) can be shown using similar ideas. Thus, both the families of states (5) and (6) are proper coherent states.

It is relevant to investigate also how the rest of the GK axioms, viz. the temporal stability of states and the action identity, can be realized in the present setting.

- (iv) The temporal stability of the states will not be fully satisfied initially, but hold only to an approximation. We seek some conditions under which the following states should belong to the same family of states

$$|\mathbf{q}, \mathbf{p}, \tau; \bar{E}\rangle = e^{-\frac{i}{\hbar}\tau H} |\mathbf{q}, \mathbf{p}; \bar{E}\rangle, \quad |\mathbf{q}, \mathbf{p}, \tau; \bar{E}\rangle_A = e^{-\frac{i}{\hbar}\tau H} |\mathbf{q}, \mathbf{p}; \bar{E}\rangle_A. \quad (12)$$

We focus again on the first type of states and rewrite that expression as

$$e^{-\frac{i}{\hbar}\tau H} |\mathbf{q}, \mathbf{p}; \bar{E}\rangle = U(\mathbf{q}, \mathbf{p}) e^{-\frac{i}{\hbar}\tau H} |\eta_{\bar{E}}\rangle + [e^{-\frac{i}{\hbar}\tau H}, U(\mathbf{q}, \mathbf{p})] |\eta_{\bar{E}}\rangle. \quad (13)$$

The evolution of the fiducial state, i.e. $e^{-\frac{i}{\hbar}\tau H} |\eta_{\bar{E}}\rangle = C \int_{E_0}^{E_1} e^{-\frac{i}{\hbar}\tau E} |\Psi_E\rangle dE$, can be decomposed using $E = \bar{E} + \delta$ such that, for small δ ,

$$e^{-\frac{i}{\hbar}\tau H} |\eta_{\bar{E}}\rangle = C e^{-\frac{i}{\hbar}\tau \bar{E}} \int_{E_0 - \bar{E}}^{E_1 - \bar{E}} \left(1 - \frac{i}{\hbar}\tau \delta + O(\delta^2)\right) |\Psi_{\bar{E} + \delta}\rangle d\delta = e^{-\frac{i}{\hbar}\tau \bar{E}} \left[|\eta_{\bar{E}}\rangle - \frac{i}{\hbar}\tau |O(\Delta E)\rangle \right], \quad (14)$$

where $\Delta E = E_1 - E_0$ and $|O(\Delta E)\rangle$ symbolizes a vector state of norm of order $O(\Delta E)$. Thus, one gets

$$e^{-\frac{i}{\hbar}\tau H} |\mathbf{q}, \mathbf{p}; \bar{E}\rangle = e^{-\frac{i}{\hbar}\tau \bar{E}} |\mathbf{q}, \mathbf{p}; \bar{E}\rangle - \frac{i}{\hbar}\tau e^{-\frac{i}{\hbar}\tau \bar{E}} U(\mathbf{q}, \mathbf{p}) |O(\Delta E)\rangle + [e^{-\frac{i}{\hbar}\tau H}, U(\mathbf{q}, \mathbf{p})] |\eta_{\bar{E}}\rangle, \quad (15)$$

where the first term is nothing but the initial coherent states up to a phase, the second term indicates a state modification depending on the window of integration ΔE and the last term involving the commutator cannot be further computed without knowing the form of the Hamiltonian.

Focusing now on the second type of states with its infinite integration range and using an analogous procedure, it can be inferred that the evolution of the coherent states will be such that the leading order term will be of the form of the initial coherent state plus other modifications. In short, what we reveal here is that the coherent states may not be temporally stable, in general, due to the appearance of the second and third contributions. However, depending on the system under investigation, this temporal stability may be satisfied at a certain order of perturbation of the parameters of the coherent states. In the following, we will discuss particular types of system for which this is indeed the case.

- (v) The action identity: Given (12), we look for a set of new canonical variables (J, γ) and (J_A, γ_A) satisfying the following relations

$$\begin{aligned} \langle \mathbf{q}, \mathbf{p}, \tau; \bar{E} | H | \mathbf{q}, \mathbf{p}, \tau; \bar{E} \rangle &= \omega J, & \dot{\gamma} &= \omega, \\ {}_A \langle \mathbf{q}, \mathbf{p}, \tau; \bar{E} | H | \mathbf{q}, \mathbf{p}, \tau; \bar{E} \rangle_A &= \omega_A J_A, & \dot{\gamma}_A &= \omega_A. \end{aligned} \quad (16)$$

By inverting $\mathbf{q}(J, \omega)$, $\mathbf{p}(J, \omega)$, $\mathbf{q}(J_A, \omega_A)$ and $\mathbf{p}(J_A, \omega_A)$, and setting $\omega = 1 = \omega_A$ from initial convention, the following identities hold

$$\langle \mathbf{q}(J), \mathbf{p}(J), \tau; \bar{E} | H | \mathbf{q}(J), \mathbf{p}(J), \tau; \bar{E} \rangle = \langle J, \gamma | H | J, \gamma \rangle = J, \quad \gamma = \tau$$

$${}_A\langle \mathbf{q}(J_A), \mathbf{p}(J_A), \tau; \bar{E} | H | \mathbf{q}(J_A), \mathbf{p}(J_A), \tau; \bar{E} \rangle_A = \langle J_A, \gamma_A | H | J_A, \gamma_A \rangle = J_A, \gamma_A = \tau. \quad (17)$$

Here $|J, \gamma\rangle$ and $|J_A, \gamma_A\rangle$ are viewed as new labels for the coherent states. This statement depends on the form of the Hamiltonian and will be discussed for particular examples.

The above basic GK ingredients (i)-(iii) will be the requirements that the coherent states built in the sequel should satisfy. We have already shown that the axiom (iv) will be only valid at a certain parameter regime whereas (v) will be checked case by case.

3 Applications

Applications of the above formal construction of coherent states are now provided. We first discuss in length the case of the free particle and, in a streamlined analysis, we study the particle in an inverted harmonic potential. The latter system has been studied for years for different purposes [14], and recently seems to have a revival due to the interest of pseudo-bosons [18][19][20]. These types of continuous-spectrum systems have non-normalizable eigenvectors, and we will illustrate how to construct coherent states for these systems.

3.1 Free particle coherent states

Spectrum and fiducial vector regularization - One of the most simple examples for which one can test the above ideas is the motion of free particle on a straight line \mathbb{R} . After canonical quantization and setting $[Q, P] = i\hbar$ for Q and P self-adjoint, the system can be described by the quantum Hamiltonian given by

$$H = \frac{1}{2m}P^2 = \frac{-\hbar^2}{2m} \frac{d^2}{dx^2}. \quad (18)$$

The Hilbert space of the system is simply given by the complex span of the position operator Q eigenbasis denoted by $|x\rangle$. H admits the following set of eigenvectors and eigenfunctions

$$\begin{aligned} H|\Psi_k\rangle &= \frac{\hbar^2 k^2}{2m} |\Psi_k\rangle, \\ \langle x|\Psi_k\rangle &= \Psi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}, \quad \langle x|H|\Psi_k\rangle = \frac{\hbar^2 k^2}{2m} \Psi_k(x). \end{aligned} \quad (19)$$

The properties of the states $|\Psi_k\rangle$ are direct: they are not normalizable and they obey

$$\langle \Psi_k | \Psi_{k'} \rangle = \delta(k' - k). \quad (20)$$

Thus, for any k , $|\Psi_k\rangle$ cannot be chosen as a fiducial vector for defining coherent states for the system.

The next stage is to define two new functions, such that for $\bar{k} \equiv \bar{k}(k_0, k_1) \in (k_0, k_1)$, $k_0 \neq k_1$, and for $A \gg 1$,

$$(a) \quad \Psi_{\bar{k}}(x) = C \int_{k_0}^{k_1} \Psi_k(x) dk = \frac{C}{i\sqrt{2\pi x}} [e^{ik_1 x} - e^{ik_0 x}],$$

$$(b) \quad \Psi_{\bar{k};A}(x) = C_A \int_{-\infty}^{\infty} e^{-A(k-\bar{k})^2} \Psi_k(x) dk = \frac{C_A}{\sqrt{2A}} e^{-\frac{x^2}{4A} + i\bar{k}x}. \quad (21)$$

For any $\bar{k} \in (k_0, k_1)$, we compute the norm of the first function (a):

$$\int_{-\infty}^{\infty} \Psi_{\bar{k}}^*(x) \Psi_{\bar{k}'}(x) dx = C^2(k_1 - k_0) = 1 \quad \Leftrightarrow \quad C = \frac{1}{\sqrt{k_1 - k_0}} \quad (22)$$

The overlap becomes a finite quantity and is normalized when C is fixed appropriately as done above. On the other hand, for the second type of state (b), we have

$$\int_{-\infty}^{\infty} (\Psi_{\bar{k};A}(x))^* \Psi_{\bar{k};A}(x) dx = C_A^2 \sqrt{\frac{\pi}{2A}} = 1 \quad \Leftrightarrow \quad C_A = \left(\frac{2A}{\pi} \right)^{\frac{1}{4}}. \quad (23)$$

Thus these states are normalized.

Two vector states $|\Psi_{\bar{k}}\rangle$ and $|\Psi_{\bar{k}}\rangle_A$ can be defined from $\Psi_{\bar{k}}(x) = \langle x | \Psi_{\bar{k}} \rangle$ and $\Psi_{\bar{k};A}(x) = \langle x | \Psi_{\bar{k}} \rangle_A$, respectively,

$$|\Psi_{\bar{k}}\rangle = C \int_{k_0}^{k_1} |\Psi_k\rangle dk, \quad |\Psi_{\bar{k}}\rangle_A = C_A \int_{-\infty}^{\infty} e^{-A(k-\bar{k})^2} |\Psi_k\rangle dk. \quad (24)$$

These will be considered as our normalized fiducial vectors.

Coherent states and GK axioms - Introducing the unitary operator $U(q, p) = e^{-\frac{i}{\hbar}qP} e^{\frac{i}{\hbar}pQ}$, $q \in \mathbb{R}$, $p \in \mathbb{R}$, we define two families of states such that

$$(a) \quad |q, p; \bar{k}\rangle = U(q, p) |\Psi_{\bar{k}}\rangle, \quad \langle x | q, p; \bar{k} \rangle = \Phi_{\bar{k}}(q, p; x) = e^{\frac{i}{\hbar}p(x-q)} \Psi_{\bar{k}}(x-q), \quad (25)$$

$$(b) \quad |q, p; \bar{k}\rangle_A = U(q, p) |\Psi_{\bar{k}}\rangle_A, \quad \langle x | q, p; \bar{k} \rangle_A = \Phi_{\bar{k};A}(q, p; x) = e^{\frac{i}{\hbar}p(x-q)} \Psi_{\bar{k};A}(x-q). \quad (26)$$

Let us check the GK axioms of coherent states for the set of states $\{|q, p; \bar{k}\rangle\}_{q,p \in \mathbb{R}}$ using the family $\{\Phi_{\bar{k}}(q, p, x)\}_{q,p \in \mathbb{R}}$.

(i) The continuity in labels q, p is obvious.

(ii) The normalizability condition can be checked since it corresponds to the condition that¹ $\Phi_{\bar{k};\bullet}(q, p, x) \in L^2(\mathbb{R}, dx)$ is of norm 1, the symbol $\bullet \in \{\emptyset, A\}$. We have

$$\bullet \langle q, p; \bar{k} | q, p; \bar{k} \rangle \bullet = \int_{-\infty}^{\infty} (\Phi_{\bar{k};\bullet}(q, p, x))^* \Phi_{\bar{k};\bullet}(q, p, x) dx = \int_{-\infty}^{\infty} (\Psi_{\bar{k};\bullet}(x-q))^* \Psi_{\bar{k};\bullet}(x-q) dx \quad (27)$$

which, by a change of variable $\tilde{x} = x - q$, reduces to the earlier calculations on the L^2 -normalizability of $\Psi_{\bar{k};\bullet}(x)$. This result can be naturally seen as a consequence of the unitarity of the operator $U(q, p)$.

¹ We will use henceforth the compact notation $|\Psi_{\bar{k};\bullet}\rangle$ for both kinds of states $\bullet \in \{\emptyset, A\}$ when the statement is valid in both situations.

(iii) The resolution of the identity has to be verified:

$$\begin{aligned} \int_{\mathbb{R}^2} \langle x|q, p; \bar{k}\rangle_{\bullet} \langle q, p; \bar{k}|x'\rangle \frac{dpdq}{2\pi\hbar} &= \int_{\mathbb{R}^2} \Phi_{\bar{k};\bullet}(q, p, x) (\Phi_{\bar{k};\bullet}(q, p, x'))^* \frac{dpdq}{2\pi\hbar} \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \Psi_{\bar{k};\bullet}(x-q) \Psi_{\bar{k};\bullet}^*(x'-q) dq \right] e^{\frac{i}{\hbar}p(x-x')} \frac{dp}{2\pi\hbar} = \delta(x-x') = \langle x|x'\rangle, \end{aligned} \quad (28)$$

where we use the fact that the integrations in q and x are similar to compute the term as performed in (27). Hence $|q, p; \bar{k}\rangle$ resolves the unity of $L^2(\mathbb{R}, dx)$.

(iv) The temporal stability condition will be satisfied only at the leading order of a small parameter depending on the two types of coherent states. However, temporal stability can be recovered for expectation values as the last step in a calculation.

Let us introduce states endowed with another evolution parameter τ such that

$$|q, p, \tau; \bar{k}\rangle_{\bullet} = e^{-\frac{i}{\hbar}\tau H} |q, p; \bar{k}\rangle_{\bullet}, \quad (29)$$

then of course

$$e^{-\frac{i}{\hbar}tH} |q, p, \tau; \bar{k}\rangle_{\bullet} = |q, p, \tau + t; \bar{k}\rangle_{\bullet}. \quad (30)$$

Hence the new state $|q, p, \tau; \bar{k}\rangle_{\bullet}$ implemented with the parameter τ is stable under time evolution. Nevertheless, we could have asked more from our coherent state and have required that without introducing another parameter, $e^{-\frac{i}{\hbar}\tau H} |q, p; \bar{k}\rangle_{\bullet}$ evolves within the same family of states $\{|q, p; \bar{k}\rangle_{\bullet}\}_{(q,p) \in \mathbb{R}^2}$. This condition should be considered as the true meaning of temporal stability under time evolution. The later statement and further consequences have been also developed in [10, 21] (and a way to generate such states considered in [22]). We seek conditions under which this statement holds.

For any coherent state described above, we have

$$e^{-\frac{i}{\hbar}\tau H} |q, p; \bar{k}\rangle_{\bullet} = e^{-\frac{i}{\hbar}qP} e^{\frac{i}{\hbar}\tau \frac{p^2}{2m}} e^{-\frac{i}{m\hbar}\tau pP} e^{\frac{i}{\hbar}pQ} |\Psi_{\bar{k}}; \tau\rangle_{\bullet} = e^{\frac{i}{\hbar}\tau \frac{p^2}{2m}} e^{-\frac{i}{\hbar}(q + \frac{p}{m}\tau)P} e^{\frac{i}{\hbar}pQ} |\Psi_{\bar{k}}; \tau\rangle_{\bullet}, \quad (31)$$

where $|\Psi_{\bar{k}}; \tau\rangle_{\bullet} = e^{-\frac{i}{\hbar}\tau H} |\Psi_{\bar{k}}\rangle_{\bullet}$.

Computing the first type of shifted state $|\Psi_{\bar{k}}; \tau\rangle$, introducing $\tilde{k}(k_1, k_0)$, a function to be specified later, $\delta = k - \tilde{k}$ and at leading orders in δ , one finds

$$\begin{aligned} \langle x|\Psi_{\bar{k}}; \tau\rangle &= e^{-i\frac{\hbar\tau}{2m}\tilde{k}^2} \langle x|\Psi_{\bar{k}}\rangle - \frac{C}{\sqrt{2\pi}} (i\tilde{k} \frac{\hbar\tau}{m}) e^{-i\frac{\hbar\tau}{2m}\tilde{k}^2} e^{i\tilde{k}x} \int_{k_0-\tilde{k}}^{k_1-\tilde{k}} \delta e^{i\delta x} d\delta \\ &+ \frac{C}{\sqrt{2\pi}} e^{-i\frac{\hbar\tau}{2m}\tilde{k}^2} e^{i\tilde{k}x} \int_{k_0-\tilde{k}}^{k_1-\tilde{k}} O(\delta^2) e^{i\delta x} d\delta. \end{aligned} \quad (32)$$

Defining $\Delta = (k_1 - k_0)/2$ and $\ell = (k_0 + k_1)/2 - \tilde{k}$, and using the new variable $\tilde{\delta} = \delta - \ell$, we can re-express the second term integral above as

$$e^{i\tilde{k}x} \int_{k_0-\tilde{k}}^{k_1-\tilde{k}} \delta e^{i\delta x} d\delta = 2ie^{i(\tilde{k}+\ell)x} \left[\frac{1}{x^2} \sin(\Delta x) - \frac{1}{x} \Delta \cos(\Delta x) - \frac{i\ell}{x} \sin(\Delta x) \right].$$

Let us consider the vector $|\Pi_\Delta\rangle$ corresponding to the last quantity and evaluate its norm as

$$\int_{-\infty}^{\infty} (\Pi_\Delta(x))^* \Pi_\Delta(x) dx = c \ell^2 \Delta + O(\Delta^2), \quad (33)$$

where c is some constant. Thus $|\Pi_\Delta\rangle$ is in the Hilbert space and its norm is of order $\Delta^{\frac{1}{2}}$. Introducing this result into the expression (32) yields

$$\langle x | \Psi_{\tilde{k}}; \tau \rangle = e^{-i \frac{\hbar \tau}{2m} \tilde{k}^2} \langle x | \Psi_{\tilde{k}} \rangle - \frac{i}{2\sqrt{\pi\Delta}} \frac{\hbar \tau \tilde{k}}{m} e^{-i \frac{\hbar \tau}{2m} \tilde{k}^2} (\langle x | \Pi_\Delta \rangle + \langle x | O(\Delta) \rangle), \quad (34)$$

where the notation $|O(\Delta^n)\rangle$ stands for a state of norm $O(\Delta^n)$. Thus all remainder states are of norm at most $O(\Delta^{\frac{1}{2}})$. By expanding further $e^{i\tau(2\tilde{k}\delta+\delta^2)}$ in δ , it can also be checked that higher order terms integrated are of order of magnitude $O(\Delta^{\frac{1}{2}+c})$, $c \geq 0$. Thus, without any further assumption, the temporal stability condition is clearly broken at leading order by the term $|\Pi_\Delta\rangle$ with order of magnitude $O(1)$ with respect to the window of integration $\sqrt{k_1 - k_0}$. Nevertheless, two specific cases of interest may occur: (1) $0 < \Delta \leq \ell \ll 1$, i.e. if the position barycenter is closed to \tilde{k} (which is the only remaining parameter in the present context and which can be tuned to fulfill that condition), then we infer that the norm of $\|C|\Pi_\Delta\rangle\|^2 = O(\ell^2) \ll O(1) = \|\Psi_{\tilde{k}}\|^2$; (2) $\ell = 0$ therefore $\|C|\Pi_\Delta\rangle\|^2 = O(\Delta) \ll O(1) = \|\Psi_{\tilde{k}}\|^2$. Thus, under these assumptions, we write in shorthand notations

$$|\Psi_{\tilde{k}}; \tau\rangle = e^{-i \frac{\hbar \tau}{2m} \tilde{k}^2} |\Psi_{\tilde{k}}\rangle + |O(\ell)\rangle \quad \text{or} \quad |\Psi_{\tilde{k}}; \tau\rangle = e^{-i \frac{\hbar \tau}{2m} \tilde{k}^2} |\Psi_{\tilde{k}}\rangle + |O(\Delta)\rangle, \quad (35)$$

such that the evolution of the coherent states is given by

$$e^{-i \frac{\hbar \tau}{2m} H} |q, p; \tilde{k}\rangle = e^{i \frac{\hbar \tau}{2m} (\frac{p^2}{2m} - \frac{\hbar^2 \tilde{k}^2}{2m})} |q + \frac{p}{m} \tau, p; \tilde{k}\rangle + (|O(\ell)\rangle \leftrightarrow |O(\Delta)\rangle) \quad (36)$$

with indeed a clear physical meaning: a given coherent state evolves within the same family of states with parameter q translated to $q + \frac{p}{m} \tau$ at order $O(\ell)$ or at order $O(\Delta)$. For the second type of coherent states, we calculate the evolution of the fiducial state in the same manner as done earlier setting for this time $\tilde{k} = \bar{k}$. We find

$$|\Psi_{\bar{k}}; \tau\rangle_A = e^{-i \frac{\hbar \tau}{2m} \bar{k}^2} |\Psi_{\bar{k}}\rangle_A + (-i \bar{k} \frac{\tau \hbar}{m}) e^{-i \frac{\hbar \tau}{2m} \bar{k}^2} \frac{1}{A} Q |\Psi_{\bar{k}}\rangle_A + \dots \quad (37)$$

A calculation establishes that the norm of the remainder state is

$$\left({}_A \langle \Psi_{\bar{k}} | (\bar{k} \frac{\tau \hbar}{m}) e^{i \frac{\hbar \tau}{2m} \bar{k}^2} \frac{1}{A} Q \right) \left((\bar{k} \frac{\tau \hbar}{m}) e^{-i \frac{\hbar \tau}{2m} \bar{k}^2} \frac{1}{A} Q |\Psi_{\bar{k}}\rangle_A \right) = \frac{1}{A} \left(\frac{\tau \hbar}{m} \bar{k} \right)^2 \quad (38)$$

which is of order $O(1/A)$. Hence, we have

$$e^{-i \frac{\hbar \tau}{2m} H} |q, p; \bar{k}\rangle_A = e^{i \frac{\hbar \tau}{2m} (\frac{p^2}{2m} - \frac{\hbar^2 \bar{k}^2}{2m})} |q + \frac{p}{m} \tau, p; \bar{k}\rangle_A + |O(A^{-\frac{1}{2}})\rangle. \quad (39)$$

In conclusion, this family of coherent states is stable under time evolution at the dominant order, for A sufficiently large.

Note that in any situation, interestingly, the inessential phase cancels exactly for $p^2 = \hbar^2 \tilde{k}^2$, namely, when the classical Hamiltonian picks the value of the quantum value $\hbar^2 \tilde{k}^2$.

- (v) The action identity axiom can be examined by evaluating the mean value of the Hamiltonian.

The coherent states of the first type provide the Hamiltonian mean value

$$\begin{aligned} \langle q, p, \tau; \bar{k} | H | q, p, \tau; \bar{k} \rangle &= \frac{1}{2m} \langle \Psi_{\bar{k}} | (P + p)^2 | \Psi_{\bar{k}} \rangle \\ &= \frac{1}{2m} \left[\frac{1}{3} C^2 \hbar^2 [k_1^3 - k_0^3] + \hbar p C^2 [k_1^2 - k_0^2] + p^2 \right] = \omega J_{\bar{k}}(p) . \end{aligned} \quad (40)$$

The goal is to find the functions $p(J, \omega)$ and $q(J, \omega)$ where J will play the role of an action variable and ω is associated with γ which will play the role of an angle variable canonically conjugate to J . This can be achieved by considering

$$\begin{aligned} p_{\pm}(J, \omega) &= -\frac{1}{2} \hbar C^2 [k_1^2 - k_0^2] \pm \sqrt{2m\omega J_{\bar{k}}(p) - \frac{1}{3} \hbar^2 C^2 [k_1^3 - k_0^3] + \frac{1}{4} [\hbar C^2 (k_1^2 - k_0^2)]^2} \\ \arctan \left(\frac{p}{q} \right) &= \omega \quad \Rightarrow \quad q_{\pm}(J, \omega) = p_{\pm}(J, \omega) \cot \omega . \end{aligned} \quad (41)$$

Hence, setting $\omega = 1$ (omitting henceforth the dependence in ω) and choosing the relevant root $p_+(J) = p_+(J, \omega = 1)$, the action identity is given by $\gamma = \tau$ and

$$\langle q_+(J), p_+(J), \tau; \bar{k} | H | q_+(J), p_+(J), \tau; \bar{k} \rangle = J . \quad (42)$$

Meanwhile, the same calculation for the second type of coherent states yields

$$\frac{\hbar^2 \bar{k}^2}{2m} + \frac{\hbar^2}{2m} \frac{1}{4A} + \frac{\hbar p \bar{k}}{m} + \frac{p^2}{2m} = \omega_A J_A \quad (43)$$

with a similar equation for q as in (41). Again, we invert p in terms of J_A and find

$$p_{\pm}(J_A, \omega_A) = -\hbar \bar{k} \pm \sqrt{2m\omega_A J_A - \hbar \frac{1}{4A}} , \quad (44)$$

such that the action identity reads, choosing $\omega_A = 1$, the root $p_+(J_A) = p_+(J_A, \omega_A = 1)$, and $\gamma = \tau$,

$$_A \langle q_+(J_A), p_+(J_A), \tau; \bar{k} | H | q_+(J_A), p_+(J_A), \tau; \bar{k} \rangle_A = J_A . \quad (45)$$

Note that, even though τ appears in both (42) and (45), J and J_A do not depend on $\tau = \gamma$ because, in these equations, that dependence explicitly vanishes (complex conjugate phases). Thus J and J_A are independent and canonically conjugated to $\gamma = \tau$.

Saturation of the uncertainty relation - The main statistical properties of the set of states $|q, p, \tau; \bar{k}\rangle$ cannot be derived since the state $|\Psi_{\bar{k}}\rangle$ does not belong in fact to the domain of Q . Nevertheless, for the second type of coherent states, one can investigate these properties. We will not undertake such a statistical analysis in this paper, however, it is significant to ask for the saturation of the Heisenberg uncertainty relation for the type of states studied here. This is the purpose of this paragraph.

After some algebra, the following identities hold

$$\begin{aligned}\langle Q \rangle &= q, & \langle Q^2 \rangle &= A + q^2, & (\Delta Q)^2 &= A, \\ \langle P \rangle &= \hbar \bar{k} + p, & \langle P^2 \rangle &= (\hbar \bar{k} + p)^2 + \frac{1}{4A} \hbar^2, & (\Delta P)^2 &= \frac{1}{4A} \hbar^2.\end{aligned}\quad (46)$$

Hence,

$$\Delta Q \Delta P = \frac{\hbar}{2} \quad (47)$$

emphasizing the fact that the coherent states defined by (26) saturate the Heisenberg uncertainty relation.

The limit $\Delta \rightarrow 0$ and $A \rightarrow \infty$ - The limits $k_1 \rightarrow k_0$ and $A \rightarrow \infty$ for the two kinds of coherent states are of interest. The construction at these limit situations rests on the following types of states:

$$\begin{aligned}(a) \quad |\Psi_{\bar{0}}\rangle &= \lim_{k_1 \rightarrow k_0} |\Psi_{\bar{k}}\rangle = \lim_{k_1 \rightarrow k_0} \frac{1}{\sqrt{k_1 - k_0}} \int_{k_0}^{k_1} |\Psi_k\rangle dk \rightarrow 0; \\ (b) \quad |\Psi_{\bar{k}}\rangle_{\infty} &= \lim_{A \rightarrow \infty} |\Psi_{\bar{k}}\rangle_A = \lim_{A \rightarrow \infty} \left(\frac{2A}{\pi} \right)^{\frac{1}{4}} \int_{-\infty}^{\infty} e^{-A(k-\bar{k})^2} |\Psi_k\rangle dk \rightarrow 0.\end{aligned}\quad (48)$$

Hence, for this limit the entire coherent state construction is vacuous, as their inner product with $\langle x|$ confirms. However this does not mean that the limits $\Delta \rightarrow 0$ and $A \rightarrow \infty$ are without interest. In fact, such limits should be performed only *after* taking expectation values. For a general operator T , the following expectations are finite

$$\lim_{k_1 \rightarrow k_0} \langle q, p; \bar{k} | T | q, p; \bar{k} \rangle, \quad \lim_{A \rightarrow \infty} {}_A \langle q, p; \bar{k} | T | q, p; \bar{k} \rangle_A. \quad (49)$$

For instance, evaluating the expectations of the identity, one finds

$$\begin{aligned}\lim_{k_1 \rightarrow k_0} \langle q, p; \bar{k} | q, p; \bar{k} \rangle &= \lim_{k_1 \rightarrow k_0} \langle \Psi_{\bar{k}} | \Psi_{\bar{k}} \rangle = 1, \\ \lim_{A \rightarrow \infty} {}_A \langle q, p; \bar{k} | q, p; \bar{k} \rangle_A &= \lim_{A \rightarrow \infty} {}_A \langle \Psi_{\bar{k}} | \Psi_{\bar{k}} \rangle_A = 1, \\ \lim_{k_1 \rightarrow k_0} \langle q, p; \bar{k} | H | q, p; \bar{k} \rangle &= \frac{p^2}{2m}, \\ \lim_{A \rightarrow \infty} {}_A \langle q, p; \bar{k} | H | q, p; \bar{k} \rangle_A &= \frac{(\hbar \bar{k} + p)^2}{2m}.\end{aligned}\quad (50)$$

3.2 Inverted harmonic oscillator coherent states

Eigenfunctions and states - Let us consider the quantum Hamiltonian

$$H = \frac{1}{2m} P^2 - \frac{1}{2} m \omega^2 Q^2 = \frac{1}{2m} (-\hbar^2 \partial_x^2 - m^2 \omega^2 x^2), \quad (51)$$

where $[Q, P] = i\hbar$. This Hamiltonian has been previously investigated from different perspectives [14][15][16]. Our goal is to show that the procedure defined above allows us to

define coherent states for this system with non-normalizable states with an infinite continuous spectrum. For simplicity in our developments, we will use notations of [14], such that $2m = 1 = \hbar = \omega$. The resulting eigenvalue problem

$$H\psi_E = E\psi_E \quad \Leftrightarrow \quad -(\partial_x^2 + \frac{1}{4}x^2)\psi_E = E\psi_E \quad (52)$$

finds solutions in terms of the parabolic cylinder functions [23]

$$\Psi_{E,1}(x) = D_{-\frac{1}{2}(1+iE)} \left(e^{\frac{1}{4}i\pi} x \right), \quad \Psi_{E,2}(x) = D_{-\frac{1}{2}(1-iE)} \left(e^{\frac{3}{4}i\pi} x \right). \quad (53)$$

Both of these eigenfunctions are non-normalizable and the system is doubly degenerate. (See the Appendix for basic properties of $D_\nu(x)$ and other facts about the Hamiltonian (51).) Without loss of generality, we will focus on $\Psi_{E,1}(x)$. The latter function can be decomposed in a linear combination (with non-trivial, energy-dependent coefficients) of a real and an imaginary part written in terms of $\psi_{E,\pm}(x) = C_\pm W(E, \pm x)$ which are again eigenfunctions of the same operator [23]. The constants C_\pm can be fixed by the orthogonality condition (proofs of the following statements can be found in [14] or in [23])

$$\int_{-\infty}^{\infty} \psi_{E,s}(x) \psi_{E',s'}(x) dx = K_{s,E} \delta_{s,s'} \delta(E - E'), \quad s, s' = \pm, \quad (54)$$

where $K_{s,E}$ is some constant. The functions $\psi_{E,s}(x)$ are however non-normalizable: for $x \gg E$, the following approximations hold [23]:

$$\begin{aligned} W(E, x) &\sim \sqrt{\frac{2\kappa}{x}} \cos \left[\frac{1}{4}x^2 + E \log x + \frac{1}{4}\pi + \phi(E) \right], \\ W(E, -x) &\sim \sqrt{\frac{2}{\kappa x}} \sin \left[\frac{1}{4}x^2 + E \log x + \frac{1}{4}\pi + \phi(E) \right], \\ \kappa &= (1 + e^{-2\pi E})^{\frac{1}{2}} - e^{-\pi E}, \quad \phi(E) = \arg \Gamma\left[\frac{1}{2} - iE\right]. \end{aligned} \quad (55)$$

In the following, we will concentrate on the solution $\psi_E(x) \equiv \psi_{E,+}(x)$, such that the following orthogonality relation will be relevant

$$\int_{-\infty}^{\infty} \psi_E(x) \psi_{E'}(x) dx = \delta(E - E'); \quad (56)$$

thus, this will fix $C_+ = C_0 = (2\pi(1 + e^{-2\pi E}))^{-\frac{1}{2}}$.

We introduce the state family $|\psi_{E,+}\rangle$ such that in the $|x\rangle$ representation we have $\langle x|\psi_{E,+}\rangle = \psi_{E,+}(x)$, $\langle \psi_E|\psi_{E'}\rangle = \delta(E - E')$ and $\langle \psi_E|\psi_E\rangle = \infty$.

Fiducial vector regularization and coherent states - The regularization of the fiducial vector can be done according to our prescription as

$$|\Psi_{\bar{E}}\rangle = C_A \int_{-\infty}^{\infty} e^{-A(E-\bar{E})^2} |\psi_E\rangle dE,$$

$$\langle x|\Psi_{\bar{E}}\rangle = C_A \int_{-\infty}^{\infty} e^{-A(E-\bar{E})^2} \psi_E(x) dx = C_A C_0 \int_{-\infty}^{\infty} e^{-A(E-\bar{E})^2} W(E, x) dE, \quad (57)$$

where C_A is a normalization constant fixed such that the state $|\Psi_{\bar{E}}\rangle$ is normalized in the same way as (23).

We can check that the set of states defined such that

$$|q, p; \bar{E}\rangle = U(q, p)|\Psi_{\bar{E}}\rangle = e^{-iqP} e^{ipQ} |\Psi_{\bar{E}}\rangle, \quad \langle x|q, p; \bar{E}\rangle = e^{ip(x-q)} \Psi_{\bar{E}}(x-q) \quad (58)$$

are coherent states.

- (i) and (ii): The continuity in labels being obvious, the normalizability condition is achieved by noticing that

$$\langle q, p; \bar{E}|q, p; \bar{E}\rangle = \langle \Psi_{\bar{E}}|\Psi_{\bar{E}}\rangle = 1. \quad (59)$$

- (iii) Let us determine the resolution of the identity by evaluating the overlap

$$\begin{aligned} \int_{\mathbb{R}^2} \langle x|q, p; \bar{E}\rangle \langle q, p; \bar{E}|x'\rangle \frac{dq dp}{2\pi\hbar} &= \int_{\mathbb{R}^2} e^{\frac{i}{\hbar}p(x-q)} \Psi_{\bar{E}}(x-q) e^{-\frac{i}{\hbar}p(x'-q)} \Psi_{\bar{E}}^*(x'-q) \frac{dq dp}{2\pi\hbar} \\ &= \delta(x-x') \int_{\mathbb{R}} \Psi_{\bar{E}}(x-q) \Psi_{\bar{E}}(x-q) dq = \langle x|x'\rangle. \end{aligned} \quad (60)$$

Thus the states $|q, p; \bar{E}\rangle$ satisfy the resolution of the identity.

- (iv) As we expect, the temporal stability of the states will only be satisfied at some approximation:

$$e^{-\frac{i}{\hbar}tH}|q, p; \bar{E}\rangle = e^{\frac{i}{\hbar}t\frac{p^2}{2m}} e^{-\frac{i}{\hbar}(q+\frac{p}{m}t)P} e^{\frac{i}{\hbar}pQ} |\Psi_{\bar{E}}; t\rangle \quad (61)$$

where we have to expand $|\Psi_{\bar{E}}; t\rangle = e^{-\frac{i}{\hbar}tH}|\Psi_{\bar{E}}\rangle$ as

$$\begin{aligned} \langle x|\Psi_{\bar{E}}; t\rangle &= C_A \int_{-\infty}^{\infty} e^{-A(E-\bar{E})^2} e^{-\frac{i}{\hbar}tE} \psi_E(x) dE = C_A \int_{-\infty}^{\infty} e^{-A\delta^2} e^{-\frac{i}{\hbar}t(\bar{E}+\delta)} \psi_{\bar{E}+\delta}(x) d\delta \\ &= C_A e^{-\frac{i}{\hbar}t\bar{E}} \int_{-\infty}^{\infty} e^{-A\delta^2} (1 - \frac{i}{\hbar}t\delta + O(\delta^2)) \psi_{\bar{E}+\delta}(x) d\delta. \end{aligned} \quad (62)$$

We aim at evaluating the norm of

$$\langle x|\Upsilon_A\rangle = \Upsilon_A(x) = C_A \frac{i}{\hbar}t \int_{-\infty}^{\infty} e^{-A\delta^2} \delta \psi_{\bar{E}+\delta}(x) d\delta. \quad (63)$$

Using the orthogonality of the functions $\psi_E(x)$, one has

$$\int_{-\infty}^{\infty} \Upsilon_A^*(x) \Upsilon_A(x) dx = C_A^2 \frac{t^2}{\hbar^2} \int_{-\infty}^{\infty} e^{-2A\delta^2} \delta^2 d\delta = \sqrt{\frac{2A}{\pi}} \frac{t^2}{\hbar^2} \frac{\sqrt{\pi}}{4\sqrt{2}A^{\frac{3}{2}}} = \frac{t^2}{4\hbar^2} \frac{1}{A}, \quad (64)$$

which implies that Υ_A is again of norm $O(1/\sqrt{A})$. Thus, as claimed, temporal stability will be obeyed at leading order and broken at $O(1/\sqrt{A})$.

- (v) The action identity requires to find (J_A, γ_A) such that, given $H = P^2 - \frac{1}{4}Q^2$ in appropriate units, we have

$$\begin{aligned}
{}_A\langle q, p, t; \bar{k} | H | q, p, t; \bar{k} \rangle_A &= {}_A\langle \Psi_{\bar{E}} | (P + p)^2 | \Psi_{\bar{E}} \rangle_A - \frac{1}{4} {}_A\langle \Psi_{\bar{E}} | (Q + q)^2 | \Psi_{\bar{E}} \rangle_A \\
&= {}_A\langle \Psi_{\bar{E}} | H | \Psi_{\bar{E}} \rangle_A + 2 {}_A\langle \Psi_{\bar{E}} | (pP - \frac{1}{4}qQ) | \Psi_{\bar{E}} \rangle_A + p^2 - \frac{1}{4}q^2 \\
&= 2\hbar p K_1(A, \bar{E}) - \frac{1}{2}q K_2(A, \bar{E}) + p^2 - \frac{1}{4}q^2 = \omega_A J_A,
\end{aligned} \tag{65}$$

where $K_{1,2}(A, \bar{E})$ are functions, yet unknown, obtained after integrations. We will use the other action-angle equation in order to fix (q, p) :

$$\begin{cases} 2\hbar p K_1(A, \bar{E}) - \frac{1}{2}q K_2(A, \bar{E}) + p^2 - \frac{1}{4}q^2 = \omega_A J_A \\ \arctan\left(\frac{p}{q}\right) = \omega_A \end{cases} \tag{66}$$

A rapid inspection shows that this system admits solutions in terms of $K_{1,2}(A, \bar{E})$ which should be analyzed in order to fix the last GK axiom.

4 Conclusion

We have addressed the construction of coherent states from non-normalizable fiducial states. The main recipe that we present here is to integrate some non-normalizable initial vector over a range of energy labels in order to define a regularized fiducial vector from which any standard procedure could apply. We have illustrated two kinds of regularization: one using a sharp cut-off and one using a smooth one. It is clear that the set of coherent states determined by the second procedure has better behaviour with respect to traditional properties of coherent states than the former set of coherent states. In particular, based on the free particle Hamiltonian example, and focusing on (i) the temporal stability axiom when the relevant parameters are optimized, and (ii) the saturation of the Heisenberg uncertainty principle, we have found that the second class satisfies these properties whereas the first class fails to obey (i) and is simply undefined for (ii). (**Remark:** This latter deficiency could be resolved by using a smoothed-out smearing function with compact support of the type belonging to the test-function space \mathcal{D} , e.g., a function proportional to $\exp[-(k_1 - k)^{-2} - (k - k_0)^{-2}]$ for $k_0 \leq k \leq k_1$ and zero otherwise; unfortunately such a weight function is analytically difficult to deal with.) The temporal stability axiom at leading order of parameters can be restored for the first class of coherent states with further assumptions. Similar remarks can be made for the second system we studied, i.e., the inverted harmonic oscillator, thereby generating a new class of coherent states adapted to this system.

If we restrict our requirements, we note that both formulations lead to coherent states in the original sense since they are normalized, continuous in their labels, and possess a traditional resolution of the identity. Furthermore, the formulation we offer here can be useful in constructing additional sets of normalizable coherent states on Hilbert spaces with fiducial vectors drawn from non-normalizable dynamical eigenvectors.

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Appendix: The inverted harmonic oscillator

We review some basic properties related with the so-called parabolic cylinder functions D_ν and the negative sign or inverted harmonic oscillator.

By definition, the functions $D_\nu(z)$ are solutions of the Weber differential equation [23][24]

$$\frac{d^2}{dz^2} f(z) + \left(\nu + \frac{1}{2} - \frac{z^2}{4}\right) f(z) = 0, \quad (\text{A.1})$$

with $z \in \mathbb{C}$ and $\nu \in \mathbb{R}$. In fact, there are two independent solutions of this second order differential equation: $D_\nu(z)$ and $D_{\nu-1}(iz)$. The general solution of (A.1) is therefore

$$f(z) = C_1 D_\nu(z) + C_2 D_{\nu-1}(iz), \quad (\text{A.2})$$

where $C_1, C_2 \in \mathbb{C}$.

For our purpose, we discuss the spectrum of the Hamiltonian of the inverted harmonic oscillator [14]:

$$H = \frac{1}{2m}(p^2 - k^2 x^2) = \frac{1}{2m}((-i\hbar)^2 \partial_x^2 - k^2 x^2) \quad (\text{A.3})$$

where $k^2 = m^2 \omega^2$, namely we want to solve the eigenvalue problem

$$H\psi_E = E\psi_E \quad \Leftrightarrow \quad \left(\partial_x^2 + \frac{k^2}{\hbar^2} x^2\right)\psi_E = -\frac{2m}{\hbar^2} E\psi_E \quad (\text{A.4})$$

with solution according to the above (A.2)

$$\begin{aligned} \Psi_E(x) &= C_1 D_{-\frac{1}{2}-i\frac{m}{\hbar k}E} \left(e^{\frac{1}{4}i\pi} \left[\frac{2k}{\hbar} \right]^{\frac{1}{2}} x \right) + C_2 D_{-\frac{1}{2}+i\frac{m}{\hbar k}E} \left(e^{\frac{3}{4}i\pi} \left[\frac{2k}{\hbar} \right]^{\frac{1}{2}} x \right) \\ &= C_1 D_{-\frac{1}{2}-i\frac{E}{\hbar\omega}} \left(e^{\frac{1}{4}i\pi} \left[\frac{2m\omega}{\hbar} \right]^{\frac{1}{2}} x \right) + C_2 D_{-\frac{1}{2}+i\frac{E}{\hbar\omega}} \left(e^{\frac{3}{4}i\pi} \left[\frac{2m\omega}{\hbar} \right]^{\frac{1}{2}} x \right). \end{aligned} \quad (\text{A.5})$$

There is another representation of $D_\nu(z)$ given in terms of confluent hypergeometric series of the first kind or Whittaker functions

$$U(\nu, z) = D_{-\frac{1}{2}-\nu}(z). \quad (\text{A.6})$$

We have the asymptotes for complex z , z large in the sense of Watson, i.e. $|z| \gg 1$ and $|\arg z| < \frac{1}{2}\pi$ [23],

$$U(\nu, z) \sim e^{-\frac{1}{4}z^2} z^{-\nu-\frac{1}{2}} \left(1 - \frac{(\nu + \frac{1}{2})(\nu + \frac{3}{2})}{2z^2} \right). \quad (\text{A.7})$$

Therefore

$$U \left(i \frac{E}{\hbar\omega}, e^{\frac{1}{4}i\pi} \left[\frac{2m\omega}{\hbar} \right]^{\frac{1}{2}} x \right) \sim K e^{-\frac{1}{4}i \left[\frac{2m\omega}{\hbar} \right] x^2} (e^{\frac{1}{4}i\pi} \left[\frac{2m\omega}{\hbar} \right]^{\frac{1}{2}} x)^{-i \frac{E}{\hbar\omega} - \frac{1}{2}}. \quad (\text{A.8})$$

Up to an overall constant, we loosely write, using the principal branch of logarithm to define complex powers which should be compatible with $|\arg e^{-\frac{1}{4}i\pi}| = \frac{\pi}{4} < \frac{1}{2}\pi$, the absolute square of the amplitude becomes

$$\left| U \left(i \frac{E}{\hbar\omega}, e^{\frac{1}{4}i\pi} \left[\frac{2m\omega}{\hbar} \right]^{\frac{1}{2}} x \right) \right|^2 \sim K |x^{-i \frac{E}{\hbar\omega} - \frac{1}{2}}|^2 = K |x|^{-1} e^{-i \frac{E}{\hbar\omega} (\ln |x| + i \arg x)} e^{+i \frac{E}{\hbar\omega} (\ln |x| - i \arg x)} \\ \sim K |x|^{-1} \quad (\text{A.9})$$

which is not integrable on the real line $x \in \mathbb{R}$. This result can be equivalently derived by the WKB method from the previous Hamiltonian problem.

Putting the Hamiltonian (A.3) in the form

$$H = \frac{-k^2}{2m} (x^2 - \frac{1}{k^2} p^2), \quad (\text{A.10})$$

we define

$$\tilde{a} = \sqrt{\frac{k}{2i\hbar}} (x + \frac{1}{k} p), \quad a = \sqrt{\frac{k}{2i\hbar}} (x - \frac{1}{k} p), \quad [a, \tilde{a}] = \frac{k}{2i\hbar} (i \frac{\hbar}{k} + i \frac{\hbar}{k}) = 1. \quad (\text{A.11})$$

Note that using the ordinary scalar product of $L^2(\mathbb{R}, dx)$, \tilde{a} is clearly not the adjoint of a . Indeed, using $x^\dagger = x$ and $p^\dagger = p$, one learns that $a^\dagger = -ia$ and so $[a, a^\dagger] = 0$. These properties lead to the so-called algebra of pseudo-bosons for which two operators obey $[a, b] = 1$ without being adjoint of one another [18][19][20]. Here the discussion is even closer to the damped harmonic oscillator in the sense of [20], i.e.,

$$H = -\frac{i\hbar\omega}{2} (\tilde{a}a + a\tilde{a}) = -i\hbar\omega (\tilde{a}a + \frac{1}{2}), \\ H^\dagger = i\hbar\omega (a^\dagger(\tilde{a})^\dagger + \frac{1}{2}) = -i\hbar\omega (a\tilde{a} - \frac{1}{2}) = H, \quad (\text{A.12})$$

as should be expected from (A.10).

References

- [1] Klauder J R 1963 *J. Math. Phys.* **4** 1058
- [2] Klauder J R and Skagerstam B-S 1985 “Coherent States” (Singapore, World Scientific)
- [3] Perelomov A M 1986 “Generalized coherent states and their applications,” (Springer, Berlin)
- [4] Klauder J R 1993 *Mod. Phys. Lett. A* **8** 1735

- [5] Klauder J R 1995 *Ann. Phys.* NY **237** 147
- [6] Klauder J R 1996 *J. Phys. A: Math. Gen.* **29** L293
- [7] Odziejewicz A 1998 *Commun. Math. Phys.* **192** 183
- [8] Gazeau J-P and Klauder J R 1999 *J. Phys. A: Math. Gen.* **32** 123
- [9] Ali S T, Antoine J-P and Gazeau J-P 2000 “Coherent States, Wavelets, and their Generalizations” (Springer-Verlag, Berlin)
- [10] Klauder J R “The current state of coherent states” Contribution to the 7th ICSSUR Conference June 2001 arXiv:quant-ph/0110108
- [11] Isham C and Klauder J R 1991 *J. Math. Phys.* **32** 607
- [12] Ben Geloun J and Klauder J R 2009 *J. Phys. A: Math. Theor.* **42** 375209
- [13] Honarasa G R, Tavassoly M K, Hatami M and Roknizadeh R 2011 *J. Phys. A: Math. Theor.* **44** 085303
- [14] Barton G 1986 *Ann. Phys.* **166** 322
- [15] Jannussis A and Skuras E 1986 *Lett. Nuovo Cimento* **94** 29
- [16] Lo C F 1990 *Phys. Rev. A* **42** 6752
- [17] Capmany J and Fernandez-Pousa C R 2011 *J. Phys. B: At. Mol. Opt. Phys.* **44** 035506
- [18] Trifonov D A 2009 in “Differential Geometry, Complex Analysis and Mathematical Physics,” eds. K. Sekigawa et al (W. Scientific, Singapore), page 241 arXiv: 0902.3744[quant-ph]
- [19] Govaerts J, Bwayi C. M. and Mattelaer O, 2009 *J. Phys. A: Math. Theor.* **42**, 445304
- [20] Bagarello F 2011 “Dissipation evidence for the quantum damped harmonic oscillator via pseudo-bosons” arXiv:1106.4638[math-ph]
- [21] Roknizadeh R and Tavassoly M K 2005 *J. Math. Phys.* **46** 445304
- [22] Yadollahi F and Tavassoly M K 2011 *Optics Communications* **284** 608
- [23] “Handbook of Mathematical Functions” 1972 10th edition *Appl. Math. Ser.* **55** Section 19 Abramowitz A and Stegun I A editors (Dover, NY)
- [24] Whittaker E T and Watson G N 1990 “A Course of Modern Analysis” 4th Ed (Cambridge University Press, Cambridge)